THE SUBMARINE FIRING PHASE DECISION

JAMES FRANCIS TUCKER

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UNITED STATES
NAVAL POSTGRADUATE SCHOOL

THESIS

SUBMARINE FIRING PHASE DECISION

BY

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Captain, United States Navy

Submitted in partial fulfillment
of the requirements
for the degree of
MASTER OF SCIENCE

United States Naval Postgraduate School
Monterey, California

1954

THE SUBMARINE FIRING

PHASE DECISION

* * * * * *

James Francis Tucker

This work is accepted as fulfilling

the thesis requirements for the degree of

MASTER OF SCIENCE

from the

United States Naval Postgraduate School

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PREFACE

As indicated in the title, this paper considers the problem of the Submarine Commanding Officer who must determine salvo size and spacing in making a torpedo attack on a target vessel.

The paper presupposes a knowledge of certain concepts of the theory of probability. The terminology and notation used in discussing mathematical probability correspond to the expressions in general usage in the literature, such as, "The Theory of Probability", M. E. Munroe, 1951. The paper is offered particularly as a consideration for possible classroom discussion by operations analysis students.

The writer wishes to thank Professor Thomas E. Oberbeck of the U. S. Naval Postgraduate School for his original suggestions, assistance, encouragement and cooperation in the preparation of this paper.

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GLOSSARY OF NOTATION

A = track angle

B = target true bearing

C = target true course

D = domain or area of uncertainty associated with position of MOT

f, g = probability density functions

h = hit probability

i = aim point designator; i = 1, 2, ---, n

L = target length

a.p.s. = aim point sequence

n = number of aim points in an aim point sequence

m = number of hits required for a specified degree of damage

= number of T/O's fired at a single aim point.

p_o = minimum acceptable probability of a hit

R = target range

S = target speed

T = time on station in days

$\sum$ = maximum salvo size

N_j = number of T/O's on board on jth day of patrol; j = 0, 1, ---, T

N_o = number of T/O's with which S/M begins patrol

F_j = expected frequence of attack on jth day of patrol

θ = angle on the bow (AOB)

$\sum_j'$ = maximum expenditure of T/O's in one attack on jth day of patrol

CHAPTER 1

ELEMENTS OF THE PROBLEM

1.1 Statement of the Problem

Each time a submarine commander decides to make an attack he must make a complete firing phase decision. This consists, among other things, of deciding:

(1)

(a) How many hits are wanted on the target.

(b) The price in torpedoes he is willing to pay for these hits.

Contained also in the firing phase decision is:

(2)

(a) Selection of the final inputs into the problem.

(b) Selection of a torpedo spread technique to insure hitting the target.

(c) Control of the firing of the torpedoes (particularly as to time).

This paper will try to give aid to (1) by an analysis of (2) based upon statistical inferences concerning 2(a) and the mathematics of probability concerning 2(b) and 2(c).

The basis for decision will be derived in terms of probability of hits obtained for torpedoes expended.

The handling of one attack is then to be correlated with the conduct of a complete patrol based upon what is desired to be accomplished by this patrol, as specified in the patrol orders.

1.2 Patrol Objectives

A patrol is considered as taking station in a geographical area under one of two alternatives as to length of stay:

1. No restrictions as to time.
2. Stay on station a specified number of days.

Each alternative is pursuant to patrol orders. The first envisages pursuing every opportunity to attack, and early expenditure of torpedoes relieves the submarine of further commitments in the area. The second reflects a need to keep the submarine in the area for a definite additional reason, (e.g. reconnaissance, rescue, or to provide a constant menace to the enemy over a specified period of time, with availability of relief, say, a governing factor).

It will be considered that submarine patrol orders contain the order to "Inflict Maximum Damage on the Enemy"[1] which may be interpreted to specify two clearly distinct tasks:

A. "Insure" kill of each target attacked,

or

B. "Insure" damage to each target attacked.

Here the word "insure" is used in a probabilistic sense, as are also the words "kill" and "damage".

It is desired that the submarine commander do his best with his opportunities in any one of <u>four</u> categories in which his orders place him, namely A1, A2, B1, B2, as shown in the following table.

1. See Theodore Roscoe, <u>Submarine Operations in World War II</u>, 1949; Sub Pac Operational Plan, p. 233.

	A	B
1	Kill each target attacked. Leave when torpedoes expended.	Damage each target attacked. Leave when torpedoes expended.
2	Space the kills. Stay on station a specified length of time.	Space the damages. Stay on station a specified length of time.

In a type 2 patrol, i.e., when A2 or B2 is assigned, note that it is desired that the commander spreads his effort over the time specified so that his ship presents, in effect, a continuous threat to the enemy. [This reasoning should also leave him with some torpedo capacity to protect himself in a tight spot throughout the duration of a type 2 patrol.]

It will be seen that in a type 2 patrol the submarine Commanding Officer must have definite knowledge of the expected frequency of enemy target traffic through his patrol area upon which to base his estimate of expected frequency of attack. This implies that there will be sources of intelligence from which he can estimate this target frequency and also that he must have some measure of his own search ability. It will be assumed that he can, from previous sources of intelligence of whatever origin, (e.g. intelligence reports, previous contacts, previous attacks, etc.) make such an estimate of the expected frequency of attack on any given day of his patrol.

1.3 Firing Phase Objective

For uniformity, we define the firing phase as the period from the time of the last observation (last set of target data inputs) until the time when the last torpedo of the salvo crosses the target track, real or extended.

In any attack, the final firing set-up, determined by whatever means, is put into the torpedo data computer at a given instant. The input data to be scrutinized here are: True bearing, range, angle on the bow(or true course) and speed of the target. True refers to the compass rose only; the input data are to be considered as a set of random variables in the probability sense. The possible errors of these input data are evaluated to predict the size and probability distribution of the location error of the target, i.e. the nature of an area (or domain) of uncertainty, say D, which is moving and expanding with elapsed time, and into which the torpedo spread is fired. In particular, it is desired to know how unit mass, representing the totality of probability of target position, described by the probability density function $f(D)$ associated with D, is distributed over this area D at a certain elapsed time t after the final inputs are made.

From knowledge of D and $f(D)$ at any given elapsed time t, it is desired to predict the aim point sequence, spread spacing, and salvo size necessary to obtain; (1) at least one hit with a minimum acceptable probability p_o when damage is wanted: or, (2) at least m hits, with a minimum acceptable probability P_o, when m hits,

on the average, are needed to sink the target.

The Commanding Officer then completes the firing phase decision on the basis of the information cited above. If a type 2 patrol is being conducted, it is also necessary to specify the salvo size for a given attack that is compatible with the requirements of the length of stay on station. The following chapters discuss the elements of this decision, i.e. salvo size as a function of the choice of one of the four patrol objectives and "how good the set-up is".

CHAPTER 2

ELEMENTS OF ONE-DIMENSIONAL CONTINUOUS PROBABILITY DISTRIBUTIONS

2.1 Scope

Certain fundamentals of probability will be discussed, using simple models, in order to develop some concepts, definitions and notation for later use.

2.2 Target Position

Consider a target T of length L on course C_o. For convenience, C_o will be taken as the positive direction on the axis of a one-dimensional coordinate system. Suppose that, at the instant t, the MOT (middle of target) is known to be on the x-axis, but its exact position is uncertain. The position of the MOT, at the instant t, may be regarded as a stochastic variable with an associated probability density function $f(x_1;t)$. The function $f(x_1;t)$ will be termed the target position density function (at time t). In the examples considered below, it will be convenient to distinguish between the cases where $f(x_1;t)$ is uniform and where $f(x_1;t)$ is normal.

Hence, when $f(x_1;t)$ is uniform, set $f(x_1;t) \equiv f_u(x_1;t)$ where

$$
(2.1) \qquad f_u(x_1;t) = \begin{cases} \dfrac{1}{Ex_1} & \text{for } a_o - \frac{1}{2}Ex_1 \leq x \leq a_o + \frac{1}{2}Ex_1 \\[2ex] 0 & \text{elsewhere} \end{cases}
$$

and a_o is the value of x_1 denoting the mean of the distribution. (See Fig. 2a). In this context, target location error is taken to be Ex_1.

When $f(x_1;t)$ is normal, set $f(x_1;t) \equiv f_n(x_1;t)$ where

$$(2.2) \qquad f_n(x_1;t) = \frac{1}{\sqrt{2\pi}\sigma x_1} \operatorname{Exp}\left(-\frac{1}{2}\left(\frac{x_1-a_o}{\sigma x_1}\right)^2\right)$$

where $-\infty < x_1 < \infty$

Again, a_o is the mean of the distribution but in this context the target location error is taken to be $2\sigma x_1$.[1]

2.3 Torpedo Position

Let x_2 represent a point on the same x-axis where a hypothetical torpedo (hereafter called T/O) crosses the axis at the instant t. Then $h(x_2;t)$, the probability of a hit on the target may be computed for any point x_2; i.e.

$$(2.3) \qquad h(x_2;t) = \int_{x_2 - \frac{1}{2}L}^{x_2 + \frac{1}{2}L} f(x_1;t)\, dx_1 .$$

The function $h(x_2;t)$ is shown in Fig. 2 b for the case where the target position error Ex_1 exceeds L. In this connection, set $Ex_1 = M \cdot L$ where M is a constant, $M > 1$.

Suppose it is known that a T/O aimed at any point on the x-axis, say x_3 will definitely cross the x-axis at the instant t, but the exact point of crossing is known only uncertainly as a stochastic variable with the associated conditional probability density function $g(x_2;t\,|x_3)$.

Then, $P(x_3;t)$, the conditional probability, of a hit on the target when the T/O is aimed at the point x_3 is given by the formula

1. Some attributes of the normal distribution are discussed in Appendix A

(2.4) $$P(x_3;t) = \int_{-\infty}^{\infty} h(x_2;t)\, g(x_2;t\,|\,x_3)\, dx_2 \,.$$

By way of illustrating $P(x_3;t)$, assume $g(x_2;t\,|\,x_3)$ is also a uniform probability density function such that:

$$g(x_2;t\,|\,x_3) = \begin{cases} \dfrac{1}{Ex_2} & \text{for } x_3 - \frac{1}{2}Ex_2 \le x_2 \le x_3 + \frac{1}{2}Ex_2 \\ 0 & \text{elsewhere} \end{cases}$$

In this context, the quantity Ex_2 will be termed the T/O error. Assume also that $f(x_1;t)$ is given by (2.1). Finally, let $Ex_1 > Ex_2 > L$, i.e., $Ex_1 = M\cdot L$, $Ex_2 = N\cdot L$, with $M > N > 1$. The graph of $P(x_3;t)$ under the assumptions of this paragraph is shown in Fig. 2 c.

Whenever the instant t is not essential to the discussion, it will be convenient to write

$$f(x_1;t) = f(x_1)$$
$$h(x_2;t) = h(x_2)$$
$$P(x_3;t) = P(x_3)$$

etc.

Since the primary interest of this chapter is the probability density function associated with target position, it will be assumed that the T/O error, Ex_2, is zero. Under this assumption,

(2.5) $$\begin{cases} x_2 = x_3 \\ P(x_3) = h(x_2) \end{cases}$$

2.4 Computation of hit probabilities.

An hypothetical example is inserted at this point, primarily as a vehicle to explain the notation used in the paper.

It is convenient to formulate first of all a system for designating the crossing points of T/O's on the x-axis. By assumption [see (2.5)] the crossing point of the T/O is the aim point of the T/O. The choice of aim points, and the probable consequences thereof is a basic objective of this paper. A definite choice of equally spaced aim points will be called an aim point sequence (a. p. s.). Inasmuch as it will be assumed throughout this paper that all T/O's cross the axis at the same instant t, the a. p. s. is not concerned with the order of firing; it specifies only the number of aim points, where aimed, and the spacing between aim points. Thus, let an a. p. s. be defined as follows:

a_o = point marking the mean of $f(x_1)$

d = 1/2 distance between adjacent aim points along the axis

n = number of aim points in the a. p. s.

See Fig. 2 d.

Then, according as n is odd or even the aim points in an a. p. s. are:

n odd	n even
$a_1 = a_o$	$a_1 = a_o \pm d$
$a_2 = a_o \pm 2d$	$a_2 = a_o \mp d$
$a_3 = a_o \mp 2d$	$a_3 = a_o \pm 3d$
	$a_4 = a_o \mp 3d$

etc.

Attention is turned first to the case where a single T/O is placed at each aim point of an a. p. s., with spacing $2d = L$, a so-called simple salvo. Later, the case of placing more than one

T/O at each aim point of an a.p.s. will be discussed.

For the case of a simple salvo, let P_i denote the probability of a hit associated with the i th aim point of an a.p.s., $i = 1, \ldots, n$.

Let $$P_1(n) = \sum_{i=1}^{n} P_i ,$$

i.e., $P_1(n)$ is the total probability of one hit for a simple salvo of n aim points.

By way of illustration, Table I gives values of P_i and $P_1(n)$ for simple salvos in two cases. The first case, designated in the table by f_u, is for $f(x_1)$ uniform with $Ex_2 = 4L$ [cf. formula (2.1)] and the second case, designated by f_n, is for $f(x_1)$ normal with $\sigma x_1 = L$ [cf. formula (2.2)]

Table I

	n = 1		n = 2		n = 3		n = 4	
a.p.	f_u	f_n	f_u	f_n	f_u	f_n	f_u	f_n
a_1	.250	.383	.250	.341	.250	.383	.250	.341
a_2			.250	.341	.250	.242	.250	.341
a_3					.250	.242	.250	.136
a_4							.250	.136
P_1	.250	.383	.500	.682	.750	.867	1.000	.954

In the case of simple salvos, with $2d = L$, note that there are no gaps nor overlaps in the spread spacing. This insures for any n, a maximum value of P_1 in the case of symmetrical target position density functions, such as $f_u(x_1)$ and $f_n(x_1)$.

Two possible criteria for the choice of n are as follows:

(a) Let p_o = minimum acceptable probability of a hit with one T/O. In this case, n = maximum i for which $p_i \geq p_o$.

(b) Let P_o minimum acceptable total probability of one hit for a. p. s. selected. In this case, n = minimum i for which $P_l \geq P_o$.

By way of illustration, suppose p_o is fixed at 0.2; then from Table I, for f_n say, it can be seen that, $n = 4$ would not be used, but instead $n = 3$ would be used. If, however, a high P_o was desired, say 0.9, then $n = 4$ would necessarily be used. The choice between (a) and (b) is a matter of preference. Hereafter, the choice of n will be given in terms of both criteria.

The possibility of more than one hit with a salvo to T/O's arises if

(1) more than one T/O is fired at each aim point of an a. p. s. with the distance between adjacent aim points exactly equal to target length, i. e., $2d = L$;

or (2) one T/O is placed at each aim point of an a. p. s. with the distance between adjacent aim points less than target length, i. e., $2d < L$;

or (3) more than one T/O is fired at each aim point and $2d$ is less than L.

These possibilities will be considered in the order just enumerated and will be referred to as Cases 1, 2, and 3, respectively.

These cases will be considered under the simplifying assumptions that

(I) each T/O crosses the x-axis exactly at its intended aim point.

(II) all T/O's of a salvo may be considered to cross the x-axis at virtually the same instant t , even if more than one T/O is fired at any aim point.

In considering Cases 1, 2, and 3, let $P_2(n, m, 2d)$ denote the probability of obtaining two hits with a salvo of size nxm in which m T/O's are placed at each of the n aim points of an a.p.s. with spacing 2d between aim points.

Under assumptions (I) and (II) above, it follows that in Case 1 there are m hits if there is one hit and hence that

$P_m(n, m, L) = P_1(n)$; that is, the value of $P_m(n, m, L)$ is identical with the value of $P_1(n)$ for a simple salvo. For example, referring to Table I, if 3 T/O's are aimed at a_1 , then three hits would be obtained with probabilities .250 and .383 , respectively, for the target position density functions f_u and f_n of Table I. Similarly, for a salvo of 6 T/O's, it is found that $P_2(3, 2, L)$ has the values .750 and .867 , respectively, for the target position density functions f_u and f_n of Table I.

Turning to Case 2, the probability of more than one hit depends in part upon the amount of "overlap" between successive aim points of the a.p.s. To illustrate, consider a salvo of two T/O's fired

in an a. p. s. of two aim points with spacing $2d = \frac{1}{2} L$. Then for a target position density function $f_n(x_1)$ with $\sigma_{x_1} = L$ (as in Table I)

$$P_1(2, 1, \tfrac{1}{2} L) = .547$$

$$P_2(2, 1, \tfrac{1}{2} L) = .197$$

Finally, in Case 3, it is clear that if m T/O's are fired at each aim point of the a. p. s. with $2d < L$, there is a possibility of m, $2m$, etc. hits. In this case it is possible to calculate $P_{jm}(n, m, 2d)$ where j is a positive integer having values 1, 2, etc. up to some maximum value depending upon $2d$. For a given salvo of $n \times m$ T/O's, it is important to note that whenever $n > 1$ and $2d < L$, a <u>penalty</u> is paid, namely, reduction in the value of $P_m(n, m, 2d)$ to achieve the probabilities $P_{jm}(n, m, 2d)$ where $j > 1$.

The spacing $2d = L$ has been measured along the x-axis, coincident with the assumed target course C_o. Consider the T/O's on any track T_o other than normal to the x-axis. Let A denote the track angle between T_o and C_o (measured as in Fig. 2 e), where for simplicity adjacent tracks are considered parallel). To intercept an interval L on the x-axis, the tracks must be spaced $L \sin A$ apart. $L \sin A$ is called the effective target length. For the one-dimensional case, it is clear that the probability of a hit associated with a given aim point does <u>not</u> depend upon the track angle. To avoid consideration of target width, very sharp track angles will not be considered here (roughly, $A < 15^o$ or $A < 165^o$ is not considered).

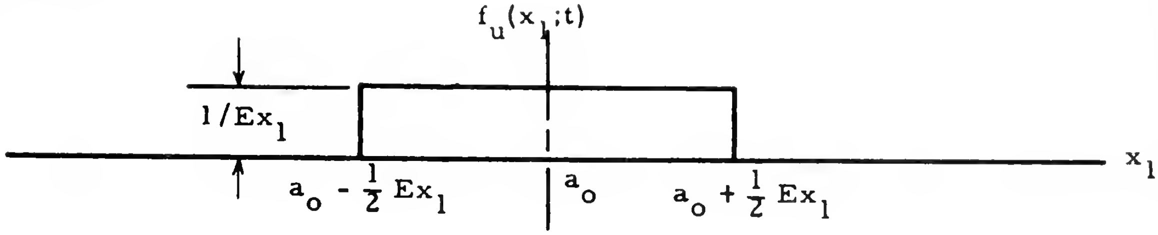


Figure 2a: $f_u(x_1;t)$

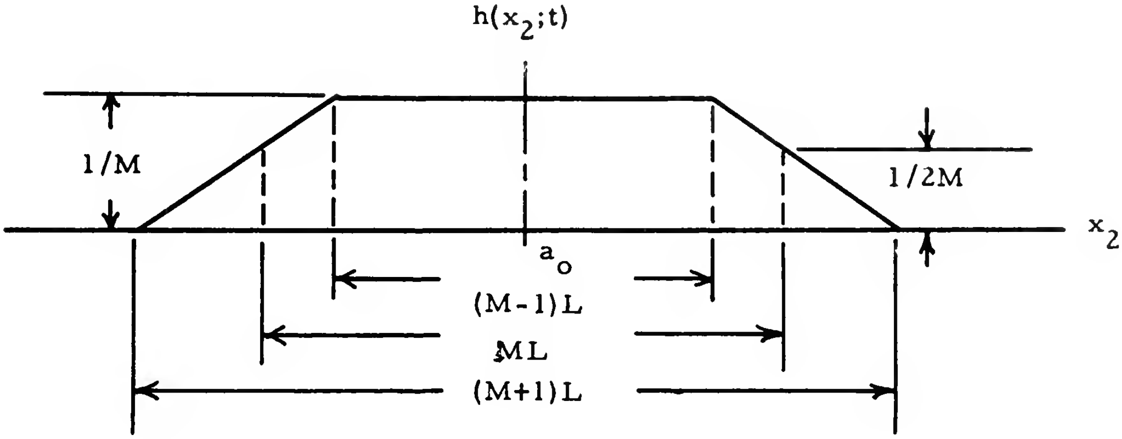


Figure 2 b: $h(x_2;t)$ where $Ex_1 = ML$, $M > 1$

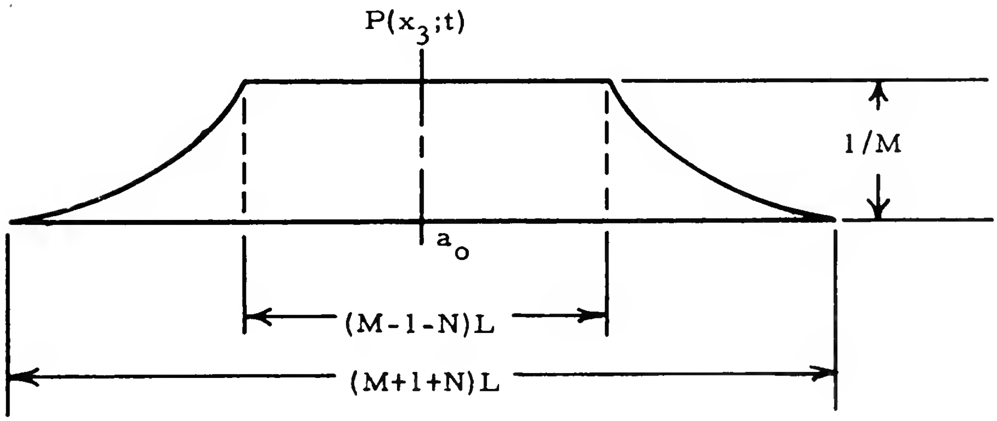


Figure 2 c: $P(x_3;t)$ where $Ex_1 = ML$, $Ex_2 = NL$, $M > N > 1$

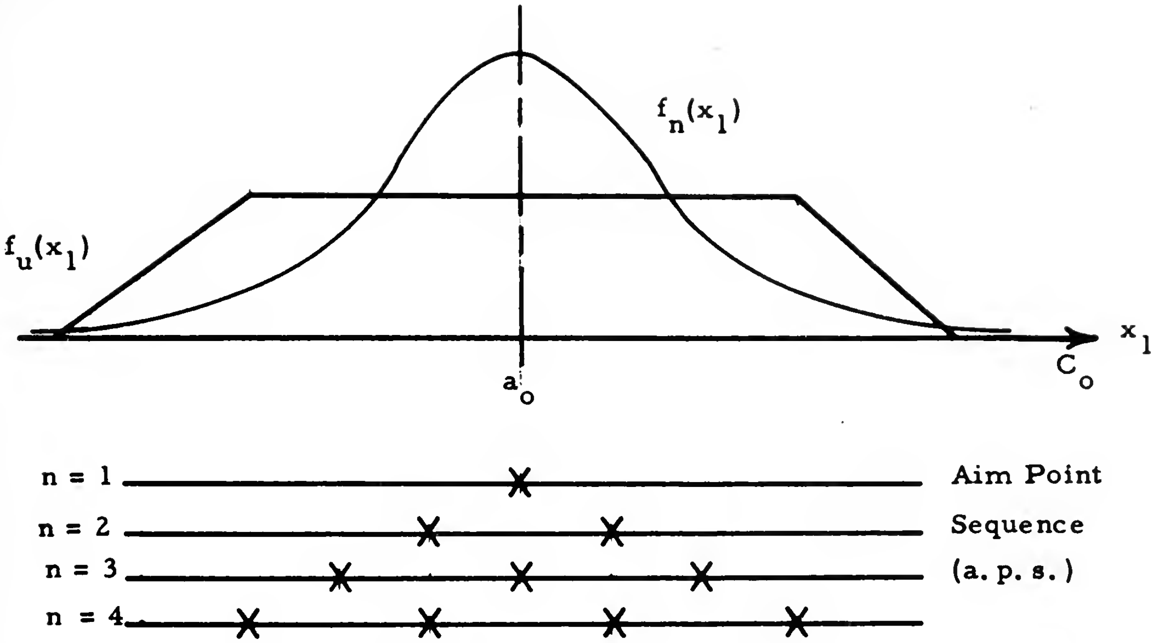


Figure 2 d: The a. p. s. depends upon n , the number of aim points, and the probability density associated with target position.

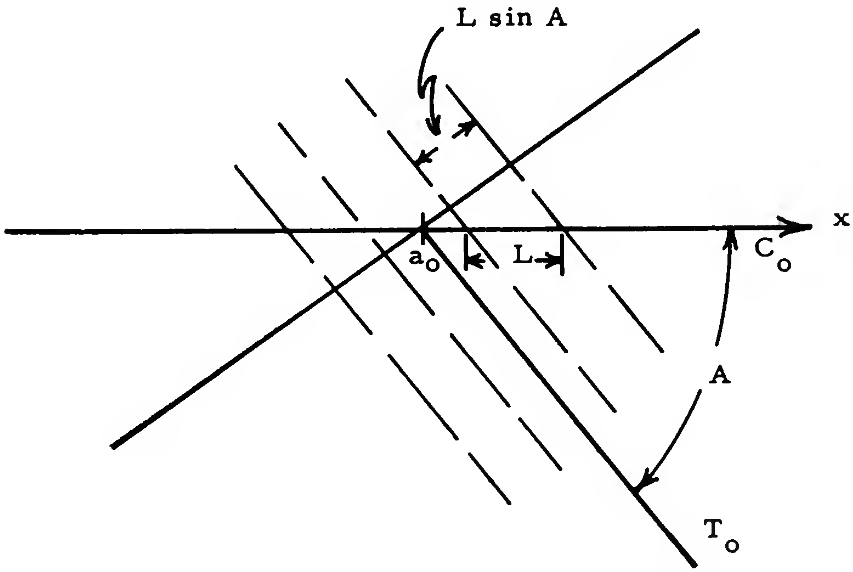


Figure 2 e: Aim point spacing along the normal to T/O track.

CHAPTER 3

ELEMENTARY PROBABILITY DENSITY FUNCTIONS FOR A STOCHASTIC VARIABLE WHICH DEPENDS LINEARLY ON TIME.

Suppose that at a certain instant of time t, measurements are made of the position and speed of a point which is moving along a straight line at constant speed. Let the position and speed measurements both be subject to error and suppose that these measurements may be regarded as observations of two independent stochastic variables. Then it is frequently necessary to "predict" the position of the point at some later time $t + \tau$. In this context, "predict" means to determine the probability density function associated with the position of the point at time $t + \tau$. The future position of the point is a stochastic variable which depends linearly on time. If X denotes the future position of the point at time $t + \tau$ and if x denotes the position of the point at time t, then

$$X = x + s\tau$$

where s is the speed.

This chapter discusses the probability density function associated with X when the probability density functions of x and s are (1) both uniform and (2) both normal.

3.1 Probability Density Functions of x and s Both Uniform.

Given: (a) A uniform probability density $f_1 \equiv f_1(x;t)$ in the interval $(x_o - \frac{1}{2}Ex, x_o + \frac{1}{2}Ex)$ associated with the

measurement of the position of the point on the line at time t.

(b) A uniform probability density $f_2 \equiv f_2(s;t)$ in the interval $(s_o - \frac{1}{2}E_s, s_o + \frac{1}{2}E_s)$ associated with the speed of the point on the line. When the position of the point at time t is known exactly and when time is considered as being measured without error, then the position of the point after an elapsed time τ is uncertain and the associated probability density function, here denoted by $\bar{f}_2 = \bar{f}_2(s\tau;t)$, is uniform in the interval $(s_o\tau - \frac{1}{2}Es\tau, s_o\tau + \frac{1}{2}Es\tau)$

Clearly at $\tau = 0$, $f_1(x;t)$ completely describes the probability density associated with the point, with mean at x_o and density $f_1 = \frac{1}{Ex}$ in the interval $(x_o - \frac{1}{2}Ex, x_o + \frac{1}{2}Ex)$ and 0 elsewhere. This would be the case at any instant t when a measurement of x is made. If no further measurement of x is available, but the position of the point at time $t_1 = t + \tau$ is required, then it is necessary to consider the new stochastic variable

$$X = x + s\tau$$

The probability density function f_3 of the new stochastic variable is obtained by the convolution of f_1 and $\bar{f}_2$.[1]

Thus,

$$\begin{aligned} f_3 = f_3(X;t_1) &= f_1(X;t) * \bar{f}_2(X;t) \\ &= \int_{-\infty}^{\infty} f_1(X - s\tau)\,\bar{f}_2(s\tau)\,\tau\,ds \\ &= \int_{-\infty}^{\infty} \bar{f}_2(X - x)\,f_1(x)\,dx \end{aligned}$$

For the simple case here, the distribution diagram for f_3 is represented in Fig. 3 a.

1. Cf: M. E. Munroe, Theory of Probability, 1951, Section 38.

It is geometrically evident that:

(1) The mean of X is moving, and at time t_1 is at $x_o + s_o\tau$.

(2) The values of X are restricted to lie within the interval

$$[x_o + s_o\tau - (\tfrac{1}{2}Ex + \tfrac{1}{2}Es\tau),\ x_o + s_o\tau + (\tfrac{1}{2}Ex + \tfrac{1}{2}Es\tau)]$$

which lengthens with τ.

(3) The area of the trapezoid representing the probability density function f_3 over this interval must always equal one (the totality of probability). Consequently, the ordinate values, which represent probability density, must necessarily be reduced as τ increases.

3.2 Probability Density Functions of x and s Both Normal.

Consider now the case where each of the density functions f_1, f_2, $\bar{f}_2$ is not uniform, but normal. In accordance with the notation in Appendix A

$$f_1 \equiv f_1(x;x_o, \sigma_x;t)$$
$$f_2 \equiv f_2(s;s_o, \sigma_s;t)$$
$$\bar{f}_2 = \bar{f}_2(s\tau;s_o\tau, \sigma_s\tau;t)$$

Then for any time $t_1 = t + \tau$, the stochastic variable $X = x + s\tau$ has a normal probability density function which, by the addition theorem for normal distributions may be represented in the notation of Appendix A

as $f_3 \equiv f_3(X;X_o, \sigma_X;t_1)$

where $X_o = x_o + s_o\tau$

$$\sigma_X^2 = \sigma_x^2 + \sigma_s^2\tau^2$$

The consequent movement of the normal variate X, the

expansion of σ_X and the thinning out of f_3 is also clear in this case as shown in Fig. 3 b.

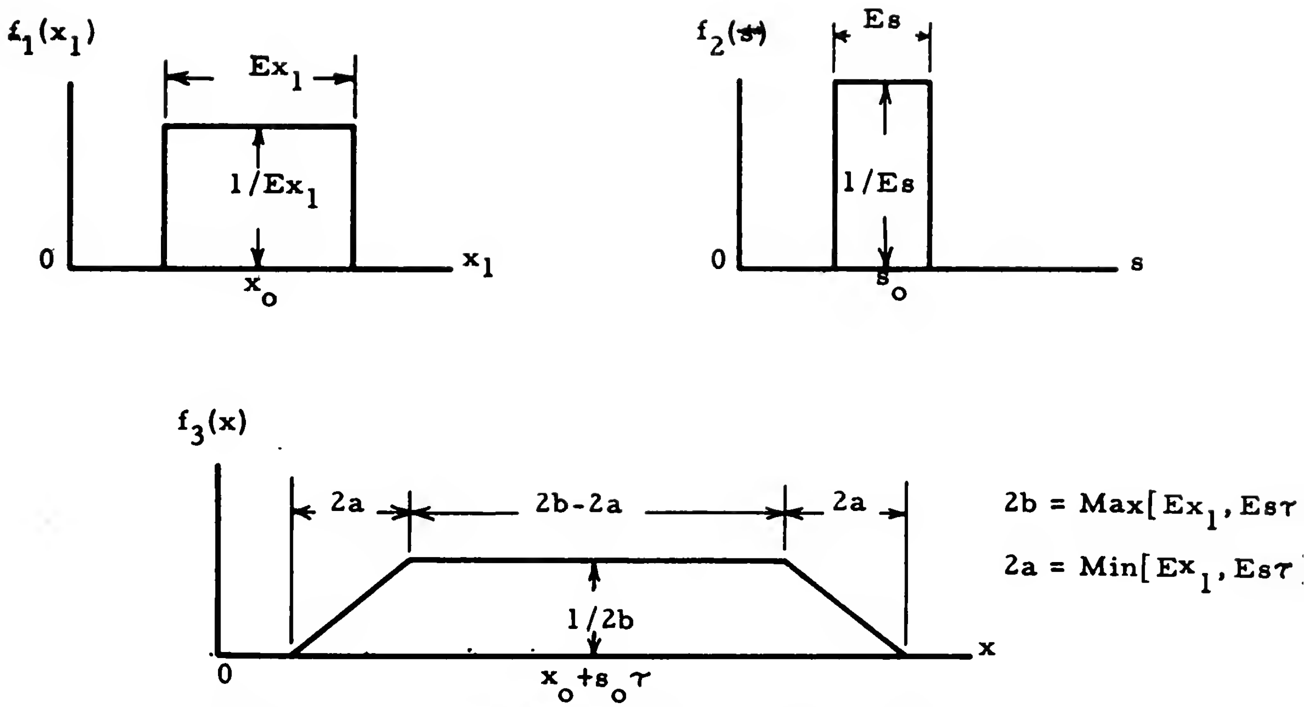


Fig. 3 a: $f_3 = f_1 * \bar{f}_2$ when f_1 and f_2 are both uniform

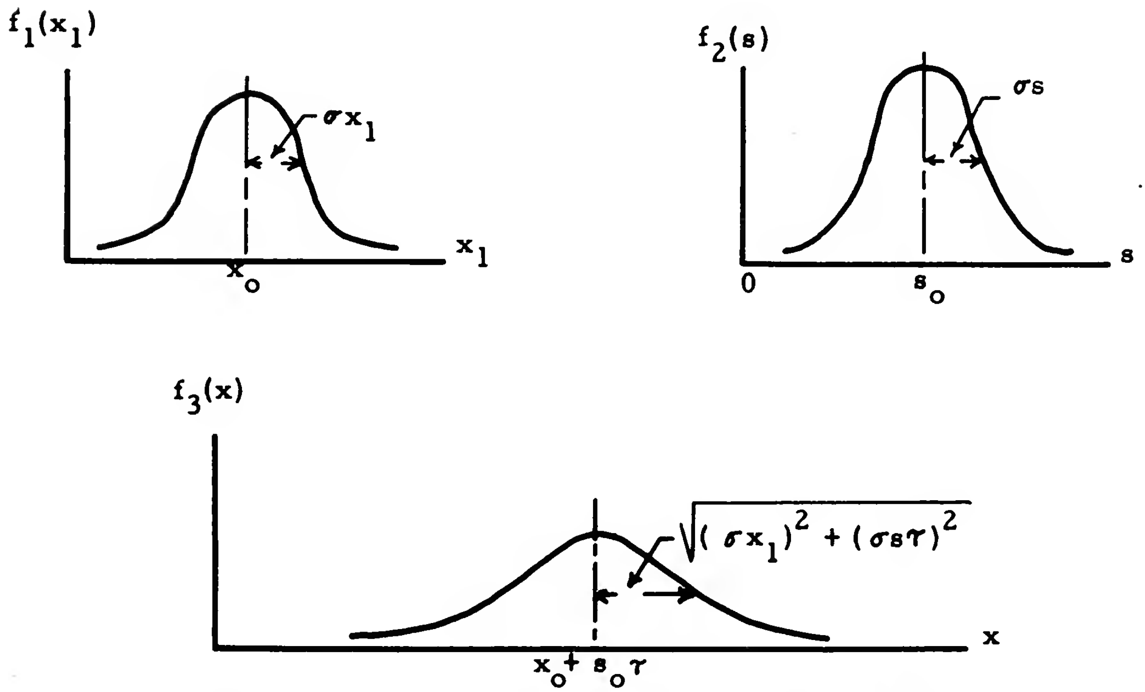


Fig. 3 b: $f_3 = f_1 * \bar{f}_2$ when f_1 and f_2 are both normal

CHAPTER 4

THE AREA OF UNCERTAINTY OF TARGET POSITION AND ITS ROLE IN THE SUBMARINE FIRING PHASE DECISION.

4.1 Preliminary Considerations and Notation

Let the independent stochastic variables (R, B) describe the position of a point in a plane; e.g., an observation of the range R and the true bearing B of a target. Let the uncertainty of the exact value of R (at time t) be associated with the uniform probability density function

$$f_1(R;t) = \begin{cases} \frac{1}{ER} & \text{for } R_o - \frac{1}{2}ER \le R \le R_o + \frac{1}{2}ER \\ 0 & \text{elsewhere} \end{cases}$$

Let B be associated with the uniform probability density function

$$f_2(B;t) = \begin{cases} \frac{1}{EB} & \text{for } B_o - \frac{1}{2}EB \le B \le B_o + \frac{1}{2}EB \\ 0 & \text{elsewhere} \end{cases}$$

For definiteness, consider B measured in radians, say from a stationary submarine whose position will be considered a fixed datum point of the problem in any coordinate system desired. The joint probability density function associated with a pair of values (R, B) at t is then

$$f_3(R, B;t) = f_1(R;t)\, f_2(B;t) .$$

Thus, f_3 is positive over the annular sector shown in Fig. 4 a as the domain $D_3 \equiv D_3(t)$.

Suppose now that no matter what point in D_3 is selected, there is a vector $\bar{\mathcal{V}}$, uncertain as to direction and speed, that can send this point $(R,B;t)$ into a new position, say $(\rho,\beta;t,\tau)$ after an elapsed time τ. As heretofore, consider τ as stop-watch time measured from the instant t without error. As shown in Fig. 4 b consider the angle θ_o that $\bar{\mathcal{V}}$ makes with B to be an independent stochastic variable, with associated probability density function

$$f_4(\theta;t) = \begin{cases} \frac{1}{E\theta} & \text{for } \theta_o - \frac{1}{2}E\theta \le \theta \le \theta_o + \frac{1}{2}E\theta \\ 0 & \text{elsewhere} \end{cases}$$

Consider the speed S, i.e., the magnitude of $\bar{\mathcal{V}}$, to be an independent stochastic variable, with an associated probability density function

$$f_5(S;t) = \begin{cases} \frac{1}{ES} & \text{for } S_o - \frac{1}{2}ES \le S \le S_o + \frac{1}{2}ES \\ 0 & \text{elsewhere} \end{cases}$$

Then at the end of a specified time τ, the probability density associated with the magnitude of the vector $\bar{\mathcal{V}}\tau$ due to the variate S is, say

$$\bar{f}_5(S\tau;t) = \begin{cases} \frac{1}{ES\tau} & \text{for } S_o - \frac{1}{2}ES\tau \le S\tau \le S_o + \frac{1}{2}ES\tau \\ 0 & \text{elsewhere} \end{cases}$$

Evidently, for <u>each</u> possible origin $(R,B;t)$ in D_3, the vector of uncertainty $\bar{\mathcal{V}}\tau$ has a domain of termination, say $D_6 = D_6(R,B;\bar{\mathcal{V}},\tau;t)$ consisting of the annular sector D_6 shown in Fig. 4 b. And with D_6 is an associated probability density function

$$f_6(S,\theta;\tau;t) = f_4(\theta;t)\, f_5(S\tau;t)$$

But $\mathcal{V}\tau$ may originate from <u>any point</u> in D_3 and further $\mathcal{V}\tau$ may assume <u>any</u> one of the set of values allowed by D_6. It may be considered that D_6 "maps" D_3 into a new domain $D_7 = D_7(\rho, \beta; t, \tau)$ where ρ and β are new stochastic variables. Functionally, D_7 may be represented as

$$(4.1) \quad D_7 = D_7 [R, f_1(R); B, f_2(B); \theta, f_4(\theta); S, f_5(S); \tau; t]$$

The region D_7 may be obtained as the result of a simple geometrical construction and is an eight-sided figure (see Fig. 4 c). The boundary of the area D_7 is comprised of the four sides of D_3 and the four sides of D_6. With D_7 is associated a positive valued probability density function $f_7 \equiv f_7(\rho, \beta; t, \tau)$. The mathematical problem is to determine exactly how unit mass, representing the totality of probability, is spread over D_7; f_7 denotes the description of this mass distribution over D_7.

No attempt to develop the analytic theory for the evaluation of the joint density function will be made here. Clearly, the probability density f_7 of a set of values of the new variates (ρ, β) at time $(t+\tau)$ is zero at the boundary edges of D_7 and rises continuously, with possibly a different slope from each edge to a plateau of maximum probability density, say

$$f_7^* = \max f_7$$

It is evident that there exists no simple way of describing the infinite variety of configurations of D_7 and the associated probability density function f_7 for D_7 and f_7 depend upon the relative magnitudes of all of the parameters indicated in formula (4.1).

For example, Fig. 4 d shows the region D_7 which results when D_3, the area of uncertainty of target position at time t , is propogated with no speed error and a discrete course error.

As a point of reference in D_7, define A_o to be the center of gravity of the mass distribution represented by f_7. The coordinates of A_o will be designated (ρ_o, β_o) or $(\rho_o, \beta_o; t, \tau)$ when the time parameters are essential. When ρ and β are independent stochastic variables then $\rho_o = \bar{\rho}$ and $\beta_o = \bar{\beta}$ where $\bar{\rho}$ and $\bar{\beta}$ denote mean values of ρ and β respectively.

Alternatively, if f_7 has a maximum value at a unique point A_o of D_7, then this point may be selected as a point of reference of D_7 and in this case the coordinates of A_o will be designated (ρ^*, β^*) where

$$f_7(\rho^*, \beta^*) = \underset{(\rho,\beta)\in D_7}{\mathrm{Max}}\, f_7(\rho, \beta) .$$

In the familiar special case where f_7 is known (or approximated) in the form

$$(4.2) \quad f_7^*(x, y) = \frac{1}{2\pi\, \sigma_x\, \sigma_y} \cdot \mathrm{Exp.}\left(- \frac{(x-x_o)^2}{2\sigma_x^2} - \frac{(y-y_o)^2}{2\sigma_y^2} \right)$$

then $A_o \equiv (x_o, y_o)$. (See Appendix A, Fig. A-2). In this special case the iso-density contours defined by $f_7^* = \epsilon$ where ϵ is a constant are concentric ellipses.

Proceeding from the formulation of D_7 and f_7 with the simplest of assumptions concerning the component stochastic variates R, B, θ, S, it would be desirable to turn to the case

where the component errors are independently and normally distributed. The reader will recognize the inherent mathematical difficulties which are beyond the scope of this paper.

4.2 The Submarine Fire Control Problem

The concepts and notation of Section 4.1 will now be related to the submarine fire control problem. Thus, let $\zeta(R, B; t)$ denote the actual range and bearing of MOT at the time of final observation of the target (instant of last inputs into Position Keeper of TDC before firing). R_o and B_o are the mean or expected values of R and B at time t with respect to the distributions $f_1(R;t)$ and $f_2(B;t)$, respectively. The functions f_1 and f_2 are assumed to be known. It will be assumed further that the values of R_o and B_o are identical with the final observations of R and B (inputs into TDC). Similarly, θ and S denote the AOB (angle on the bow) and the speed of the target at time t. θ_o and S_o are the mean or expected values of θ and S at time t with respect to the functions $f_4(\theta;t)$ and $f_5(S;t)$ respectively. Again, the functions f_4 and f_5 are assumed to be known and the values of θ_o and S_o are assumed to be identical with the final observations of θ and S (inputs into TDC). Then the region D_3 may be called the area of uncertainty of MOT at the time of last observation of the target. The region D_7 then represents where the MOT may be found at time $t+\tau$ (τ units of time after the last observation) when there are uncertainties in target range, bearing, course, and speed. The region D_7 will be

termed the area of uncertainty of MOT at $t+\tau$.

The interval τ may be considered as the sum of two intervals τ_1 and τ_2, $\qquad \tau = \tau_1 + \tau_2$

The interval τ_1, represents the "dead-time" between the last observation of the target and the instant of firing; the interval τ_2, which is a function of τ_1 is the time required for a "perfect" T/O to reach $a_o \equiv (\rho_o, \beta_o; t, \tau)$ when the T/O is fired at the instant $t+\tau_1$. Briefly, τ_2 is T/O running time and $\tau_2 = \tau_2(\tau_1)$

The submarine is assumed to have a TDC which can predict τ_2 within the interval τ_1, i.e., T/O running time is calculated before the T/O must be fired.

To this point, the direction of the target's motion has been discussed in terms of θ, the AOB. It will be convenient to speak now of the target's course C, where, of course, $C = C(\theta, B)$ and in particular $C_o = C(\theta_o, B_o)$. Inasmuch as C_o is the target's course as it appears in the TDC in accordance with previous assumptions, it is convenient to consider the aim points of the aim point sequence (see Ch. 2) of the T/O salvo to be along the course line C_o extending through the point a_o.

Unlike the simple one-dimensional case discussed in Chapter 2, there is in the two-dimensional problem no simple way to calculate the probability of a hit P_i associated with the <u>ith</u> aim point of the a. p. s. when f_7 is generated by f_1, f_2, f_4, and f_5. To discuss the two-dimensional case further, it is

convenient to use the simplest of continuous probability density functions as a model for f_7.

To simplify them, consider

$$f_7 \equiv f(x,y) = \begin{cases} \dfrac{1}{ExEy} & \text{for } x_o - \frac{1}{2}Ex \leq x \leq x_o + \frac{1}{2}Ex \\ & \quad\;\; y_o - \frac{1}{2}Ey \leq y \leq y_o + \frac{1}{2}Ey \\ 0 & \text{elsewhere} \end{cases}$$

With this simplification, $D_7(\rho,\beta;t,\tau)$ is a rectangle which will be designated $D(t+\tau)$. See Fig. 4 e.

Recall that D represents the area of uncertainty of position of MOT at time $t+\tau$. The time interval is computed for the collision instant of a "perfect" T/O with the point a_o. Let C_o, the "most likely" target course through a_o, be identical with the x-axis of coordinates. Let the collision instant $t+\tau = t_o$. To simplify further, assume that all T/O's have infinite speed through $D(t_o)$, and that all T/O's of a salvo pass through $D(t_o)$ at the same instant t_o. The width of the target will be ignored. For T/O's traversing paths parallel to the y-axis, the probability for a hit at a given aim point X_2 along C_o would be

$$h(x_2;t_o) = \int_{x_2 - \frac{1}{2}L}^{x_2 + \frac{1}{2}L} \int_{-\infty}^{\infty} f(x,y)\,dydx$$

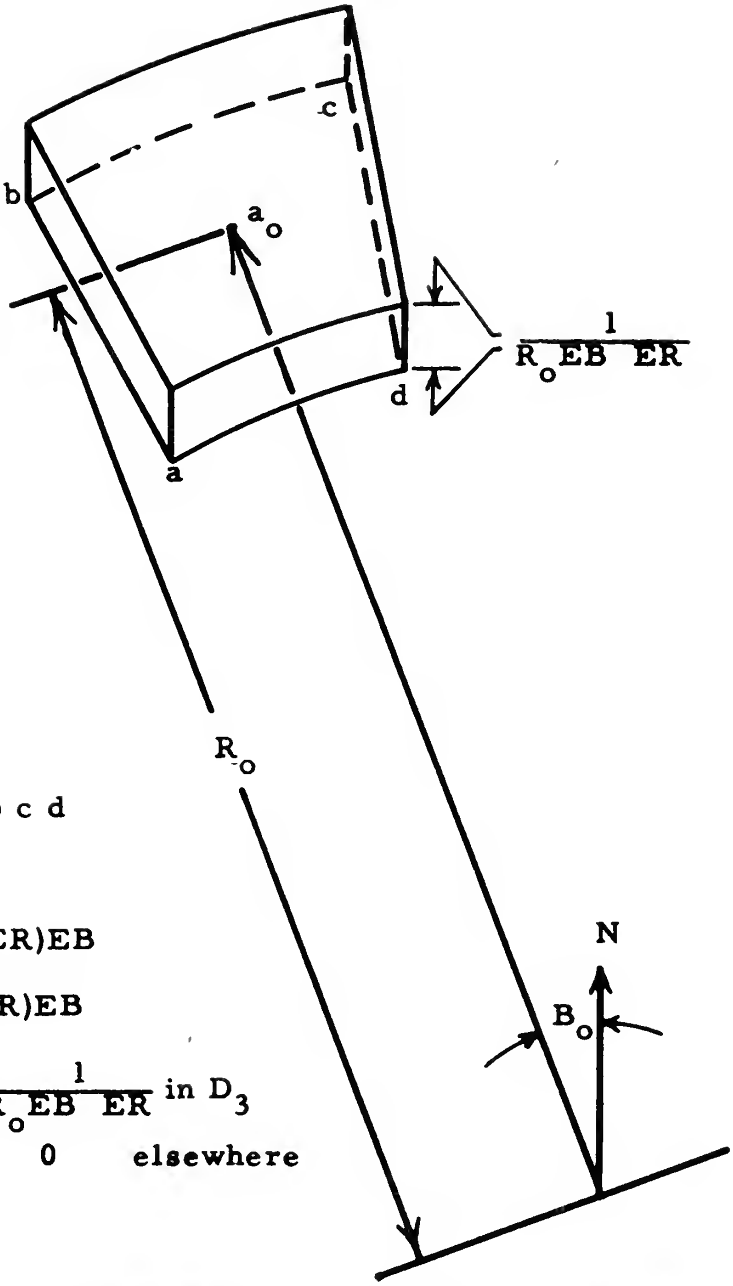
which reduces to the result (2.2) for the one-dimensional case of Chapter 2. For track angles other than 90^o this is not the case. Referring to Fig. 4 f, it will be seen that the probability of a hit for such a track is simply the volume under $f(x,y)$ "swept out" by the T/O "path" of width $L \sin A$.

Consider now the assumption that the T/O has infinite speed in the light of the simple example just discussed. Taking into account the finite speed of the T/O, a very precise calculation of probability of a hit would require determination of the earliest possible instant, say $(t_o - \delta_1)$ after firing at which the T/O enters a region $D(t_o - \delta_1)$ in which $f(t_o - \delta_1)$ has a positive value and the latest possible instant, say $(t_o + \delta_2)$ at which the T/O leaves a region $D(t_o + \delta_2)$ in which $f(t_o + \delta_2)$ has a positive value. When $f(t_o - \delta_1)$ and $f(t_o + \delta_2)$ differ sufficiently over the "path swept out" by the T/O, there may be an appreciable error in assuming that the T/O passes with infinite speed through a region $D(t_o)$ and sweeps out a path under the probability density function $f(t_o)$. Moreover, the statements above that the T/O enters $D(t_o - \delta_1)$ and leaves $D(t_o + \delta_2)$ imply that these regions have certain boundaries which will not be true when f has positive values over the entire plane as, for example, when at least one of the component probability density functions f_1, f_2, f_4, or f_5 is normal. Under these circumstances it may be convenient to consider a region, say $D(t_o - \delta_1)$ bounded by an ϵ-valued iso-density contour of $f(t_o - \delta_1)$ (cf. remarks following formula (3.2)) which the T/O enters at $t_o - \delta_1$ and another region, say $D(t_o + \delta_2)$, bounded by an ϵ-valued iso-density contour of $f(t_o + \delta_2)$ which the T/O leaves at $t_o + \delta_2$ and determine whether $f(t_o - \delta_1)$ and $f(t_o + \delta_2)$ differ "significantly" over the path swept out by the T/O. When these two functions

do not differ significantly, then the probability of a hit associated with an aim point may be visualized as the volume under $f(t_o;\epsilon)$ swept out by the path of the T/O through $D(t_o;\epsilon)$.

In actual practice, the probability density functions f_1, f_2, f_4, and f_5 associated with target range, bearing, course, and speed inputs may not be known. However, it seems evident from the analysis of this paper that these functions are essential operational data which are necessary for the submarine fire control problem if there is to be a rational basis for T/O salvos. It is likely that all four functions are normal density functions with standard deviations which may be estimated by operational trials.

Appendix B gives a brief discussion of the numerical approximation of f_7 by high speed digital computers when f_1, f_2, f_4, and f_5 are given.



$$D_3(R, B; t) = a\ b\ c\ d$$

$$a\ b = c\ d = ER$$

$$a\ d = (R_o - \tfrac{1}{2}\,ER)EB$$

$$b\ c = (R_o + \tfrac{1}{2}\,ER)EB$$

$$f_3(R, B; t) = \begin{cases} \dfrac{1}{R_o EB\ ER} & \text{in } D_3 \\ 0 & \text{elsewhere} \end{cases}$$

Figure 4 a: The annular sector $D_3(t)$ and the associated probability density function $f_3(R, B; t)$

$D_6 = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$

$P = (s_o - \frac{1}{2} Es)\tau \; E\theta$

$Q = (s_o + \frac{1}{2} Es)\tau \; E\theta$

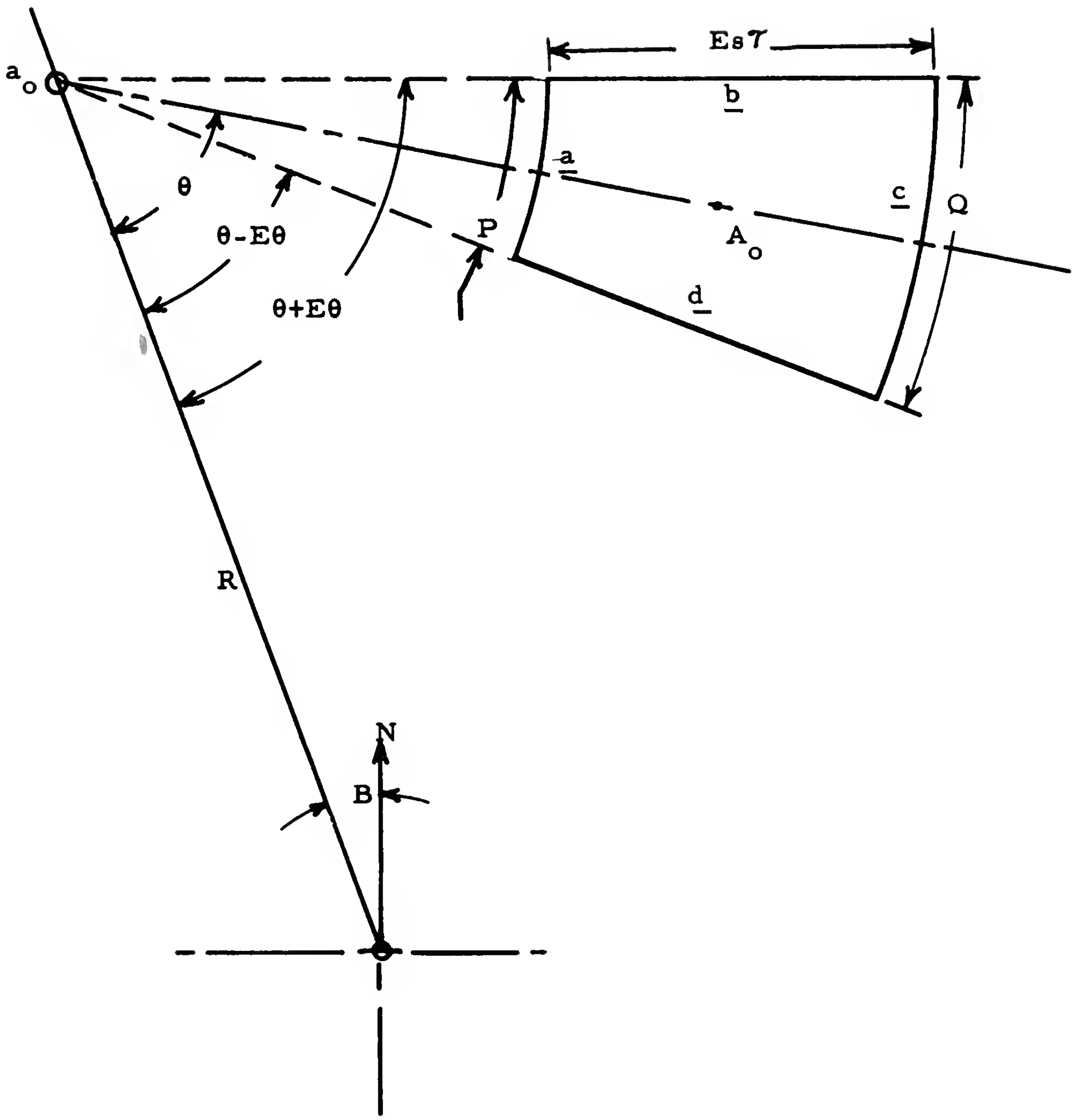


Figure 4 b: The annular sector $D_6(R, B; \bar{v}, \tau; t)$

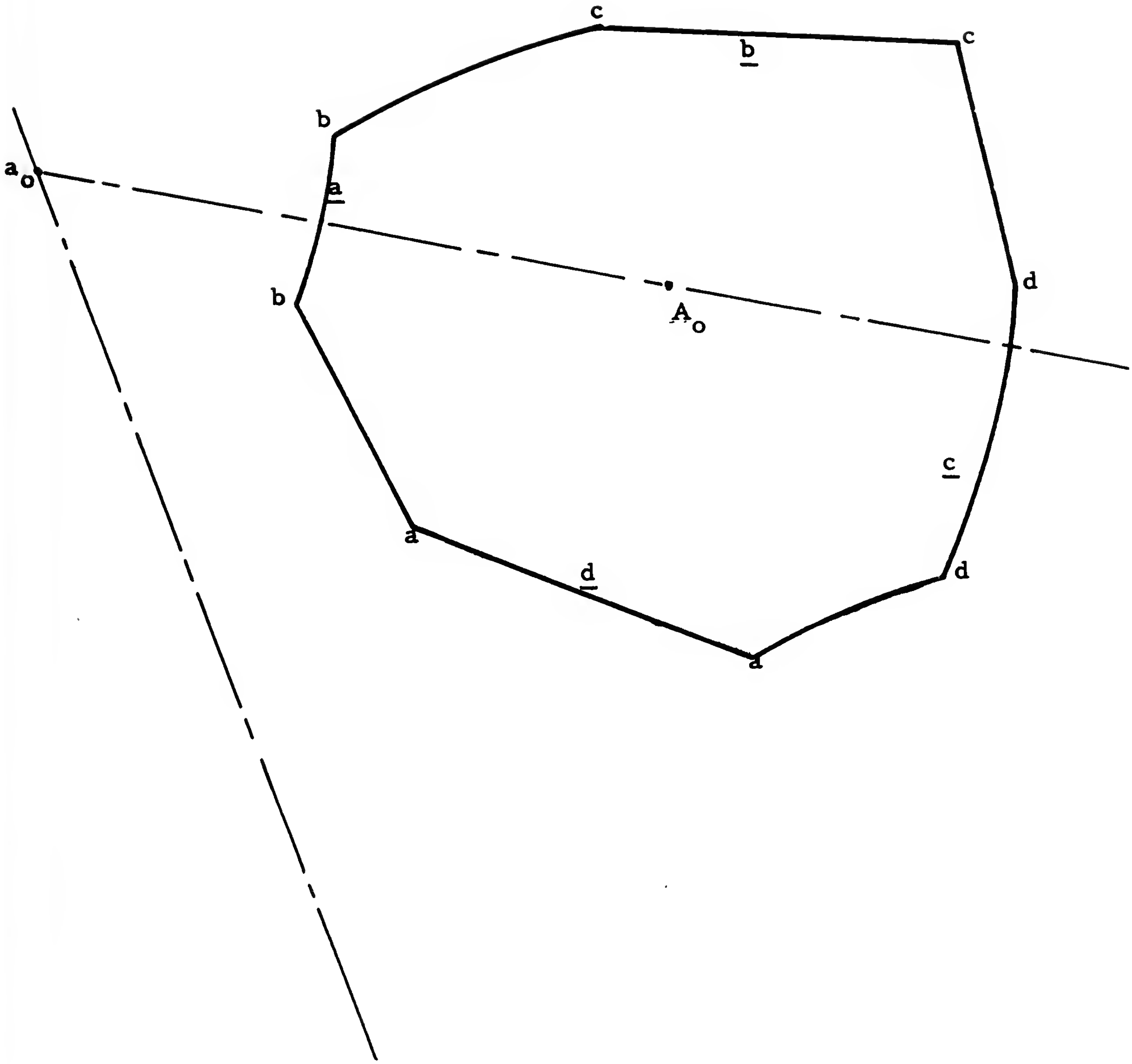
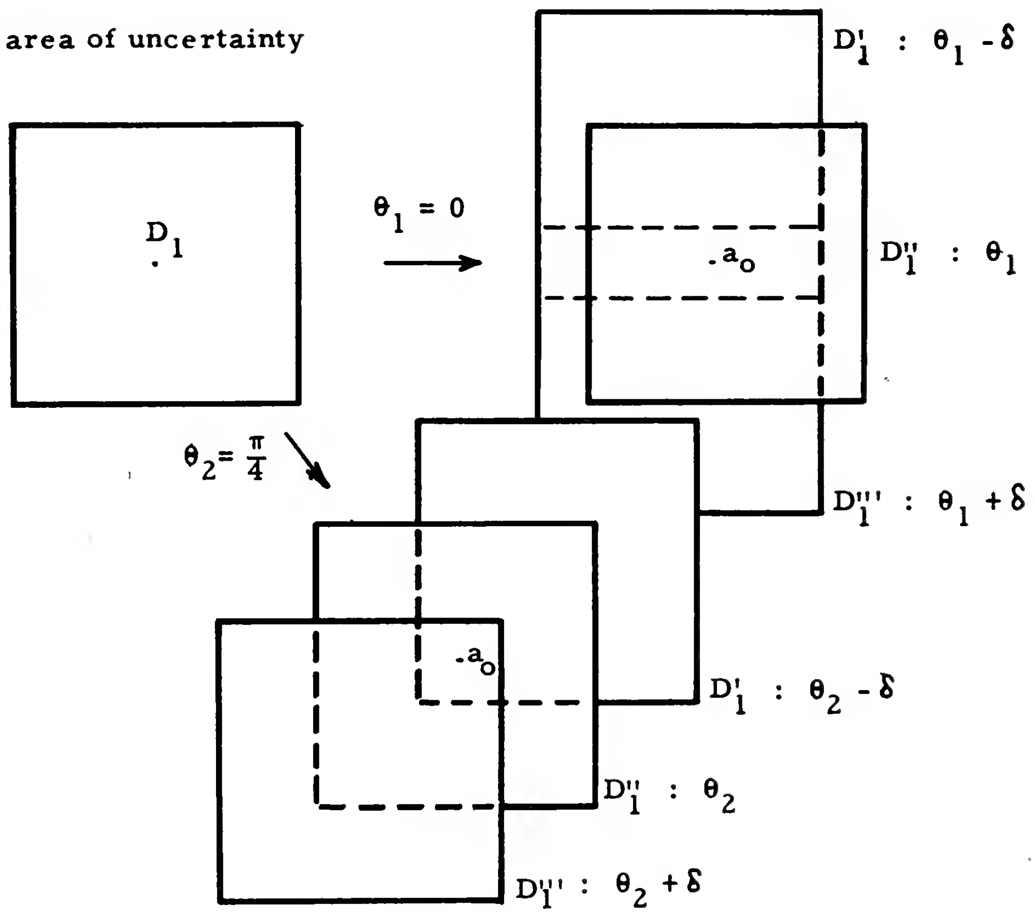


Figure 4 c: The region D_7

Figure 4 d: The movement of D_1 to D_1' , D_1'' , D_1''' at fixed speed with no speed error for 3 discrete values of θ , namely, $\theta - \delta$, θ , $\theta + \delta$ for $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{4}$.

$D_1 \equiv$ original area of uncertainty



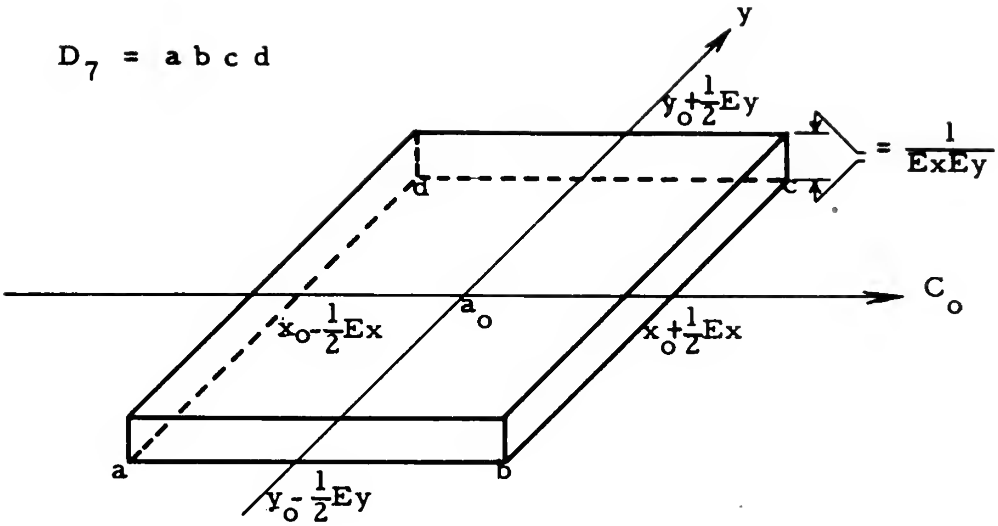


Figure 4 e: The case where D_7 is a rectangle

$$f(D) = \begin{cases} \frac{1}{ExEy} & \text{in } D_7 \\ 0 & \text{elsewhere} \end{cases}$$

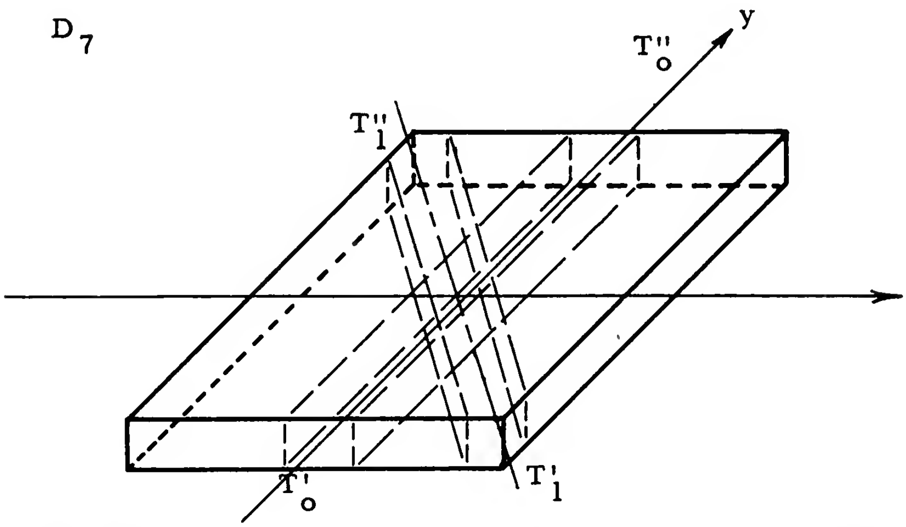


Figure 4 f: The effect of T/O track angle on the area of D_7 which is "swept out".

CHAPTER 5

GENERAL REMARKS ON TORPEDO SALVOS

5.1 Torpedo Lethality

The number of T/O's required, on the average, to destroy a target, depends on two primary factors:

(1) The accuracy of the T/O, i.e. the probability the T/O will hit the target, and

(2) The destructiveness of the T/O, i.e. the probability that the target is destroyed if the T/O hits it.

Here (1) is reduced to the problem of finding the P_i's , thence n .

For (2) in many cases a single hit is not enough to destroy a target. Even for merchant ships a single T/O hit is not usually enough to sink it, and heavy combatant ships are designed to withstand many T/O hits. To treat such cases exactly, one should determine the probabilities, say, d_1 , d_2 , d_3 , etc. of sinking the ship if the ship is hit 1, 2, 3, etc., times. Then if, for a given method of aiming T/O's, the probabilities of 1, 2, 3, etc. hits are P_1 , P_2 , P_3 , etc, the probability of sinking the ship would be $d_1P_1 + d_2P_2 + d_3P_3 + ---$, where the d's would represent damage coefficients.

For this paper it is simply assumed that there is a number m for each type ship such that, on the average, m hits of that type will sink the ship with probability $P_o = P(m, n, d) = \sum_{i=1}^{n} P_i$,

with $d = \frac{1}{2} L \sin A$, the spacing along the normal to T/O tracks for the a. p. s. described in Chapter 2.

5.2 Time Intervals in Salvo Fire

Recall that, in firing torpedoes from a submarine, the firing is not a true spread in the sense that all T/O's are launched at once to provide a simultaneous menace to the enemy. T/O's are fired singly and consecutively with a prescribed time delay to avoid countermining.

Let $\Delta\Upsilon$ = time delay (an interval of several secs.). Consider a spread of nxm T/O's, let $m = 1$. Let Υ_r be initial make ready time after last inputs to TDC at time t. The mean T/O would go at time $\Upsilon_r + \frac{n-1}{2}\Delta\Upsilon$ after time t, on an expected run time $= \Upsilon_2$ say. Let mean $\Upsilon_1 = \Upsilon_r + \frac{n-1}{2}\Delta\Upsilon$; then mean $\Upsilon = \Upsilon_1 + \Upsilon_2$ and $t_o = t + \Upsilon$.

The mean track To, commenced at time $t + \Upsilon_1$ may be considered as the datum line about which the spread, or aim point sequence (a. p. s.) is placed.

CHAPTER 6

THE FIRING PHASE DECISION

6.1 The Cardinal Rule

The number of aim points n in an a.p.s. may be regarded as depending upon the function $f(t_o; \epsilon)$ introduced in Section 4.2. Hence n is a function of Υ.

Therefore, as a cardinal rule to minimize the spreading and thinning out of probability density, avoid protracted firings:

FIRE AS SOON AS POSSIBLE AFTER DECISION TO COMMENCE FIRING PHASE.

6.2 The General Procedure

Most of the notation used here has already been discussed, otherwise it is defined in the glossary. Four sets of procedures are outlined, following the dictates of the four objectives proposed in Chapter 1. In each case n = maximum i for which $P_i \geq p_o$ and $d = \frac{1}{2} L \sin A$ along the normal to T/O tracks in the aim point sequence. Then $P_o = \sum_{i=1}^{n} P_i$

I. Fire Salvos to maximize probability of at least one hit (i.e. case B1). Do not fire T/O's at aim points for which associated probability of hit is less than p_o. As nearly as possible, all T/O's are to be expended in not more than T days; obtain extra hits as indicated by frequency of attack.

(a) If $\sum \leq n$ fire one T/O at each of first $\sum$ aim points (a.p.) (cf. Section 2.3)

(b) If $n < \Sigma$ look at Σ_j' then if:

(i) $\Sigma \leq \Sigma_j'$ fire μ T/O's at each of first n a.p. and one T/O at each of first $\Sigma - \mu n$ a.p.

(ii) $n \leq \Sigma_j' < \Sigma$, fire μ T/O's at each of first n a.p.

(iii) $\Sigma_j' < n < \Sigma$ fire one T/O at each of first n a.p.

(c) When $\Sigma_j' = 2\Sigma$ request permission to reduce value of p_o

(d) Return to base when all T/O's are expended, but do not remain on station more than T days.

II. Fire T/O's to maximize probability of m hits on targets requiring m hits to be sunk with probability P_o (i.e., case A1).

Do not fire at aim points for which associated probability of hit is less than p_o.

As nearly as possible, all T/O's are to be expended in not more than T days; obtain extra hits as indicated by frequency of attack.

(a) If $mn \geq \Sigma$, fire m T/O's at each of first ν a.p. (Optional to fire $\Sigma - m\nu$ T/O's at $(\nu + 1)$th a.p.).

(b) If $mn < \Sigma$, determine Σ_j' then if:

(i) $\Sigma \leq \Sigma_j'$ fire μ T/O's at each of first n a.p.

(ii) $mn < \Sigma_j' < \Sigma$ fire μ' T/O's at each of first n a.p.

(iii) $\Sigma_j' \leq mn < \Sigma$ fire m T/O's at each of first n a.p.

(c) When $\Sigma_j' = 2\Sigma$ request permission to increase value of P_o and/or to reduce value of p_o.

(d) Return to base when all T/O's are expended, but do not remain on station more than T_7 days.

III. <u>Remain on station for approximately T_1 days, conducting as many attacks as possible.</u> <u>Fire salvos to maximize probability of at least one hit</u> (i.e., case B2)

Do not fire T/O's at aim points for which associated probability is less than p_o. As nearly as possible, expend all T/O's by end of patrol; obtain extra hits as indicated by frequency of attack.

(a) If $\Sigma_1 \leq n$ fire one T/O at each of first Σ_1 a.p.

(b) If $n < \Sigma_1$ fire μ_1 T/O's at each of first n a.p. and one T/O at each of first $\Sigma_1 - \mu_1 n$ a.p.

(c) When $\Sigma_j' = 2\Sigma$ request permission to reduce value of p_o.

(d) Return to base when all T/O's are expended, but make this as close as possible to T_1 days.

IV. <u>Remain on station for approximately T_1 days, conducting as many attacks as possible.</u> <u>Fire salvos to maximize probability of m hits on targets requiring m hits to sink with probability P_o</u> (i.e., case A2).

Do not fire T/O's at aim points for which associated probability of hits is less than p_o. As nearly as possible expend all T/O's by the end of the patrol; obtain extra hits as indicated by frequency of attack.

(a) If $\Sigma_1 \leq mn$ fire m T/O's at each of first ν_1 a.p. (Optional to fire $\Sigma_1 - m\nu_1$ T/O's at $(\nu_1 + 1)$ <u>th</u> a.p.).

(b) If $mn < \Sigma_1$ fire μ_1 T/O's at each of first n a.p. (Optional to fire $\Sigma_1 - \mu_1 n$ T/O's at $(n + 1)$ th a.p.)

(c) When $\Sigma_j' = 2\Sigma$ request permission to increase value of P_o and/or reduce value of p_o.

(d) Return to base when all T/O's are expended, but make this as close as possible to T_1 days.

Appendix A

The Normal Distribution

A-1 The One-Dimensional case.

The normal distribution is one of the most significant mathematical models, from both the theoretical and practical standpoints, upon which the techniques of applied statistics are founded.

Using as an example, say, a set of n measurements, with errors $x_i (i = 1, 2, ---, n)$, some underlying assumptions of the distribution are;

(a) In performing the set of measurements, not all the errors x_i are equally likely to occur.

(b) In general, large errors are less likely to occur than small ones.

(c) If there is a best value m , then positive and negative errors about this value are equally likely to occur.

(d) If (c) is true, then m is the arithmetic average of the individual measurements.

The density of this distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \text{ Exp}\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)$$

Thus, a normal distribution is completely determined by specifying its mean m , and standard deviation σ i.e. $f(x;m,\sigma)$. The function is shown in Fig. A-I for several values of σ . Changing m merely shifts the curves to the right or left without changing their shape.

By the linear transformation $y = \frac{x-m}{\sigma}$, $f(x)$ is put in standard form, with $m = o$ and $\sigma = 1$. The density function is then

$$f(x;o,1) = \frac{1}{\sqrt{2\pi}} \text{Exp} \left(- \frac{1}{2} x^2 \right)$$

It is in this form that one usually finds the integral tabulated for any value of x .

A-2 The Two-Dimensional Case.

A simple case of two independent, normally distributed variables will be discussed here. Given $f_1(x;x_o,\sigma_x)$ and $f_2(y;y_o,\sigma_y)$ such that $f(x,y) = f_1(x)\, f_2(y)$. Let the point $(x_o, y_o) = a_o$.

The joint density is given by:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \text{Exp} \left\{ - \frac{1}{2} \left[\left(\frac{x-a_o}{\sigma_x} \right)^2 + \left(\frac{y-a_o}{\sigma_y} \right)^2 \right] \right\}$$

(i.e. unit mass is spread over a symmetrical mound). Fig. A-2.

Curves formed by a vertical slice parallel to an axis are of the same shape as the normal distribution curve.

The curve formed by a horizontal slice is an ellipse

$$f(x,y) = \left[\left(\frac{x-a_o}{\sigma_x} \right)^2 + \left(\frac{y-a_o}{\sigma_y} \right)^2 \right] = c^2 \text{ (a constant)}$$

The equation represents an ellipse with center at a_o semi-axes of length $(c\sigma_x, c\sigma_y)$ and axis parallel to (x,y) axes. An ellipse of this nature is called an iso-density contour ellipse.

The probability that (x,y) lies inside the ellipse equals P when $c^2 = \chi^2_p$, $f = 2$ (where c^2 is found using the x, y values and P is the probability in per cent for the corresponding fractile of the Chi-squared distribution with 2 degrees of freedom).

To define a bound for the area D covered by f(x, y), any contour ellipse ϵ may be selected, e.g. noting that:

$$\chi^2_{90} = 4.61 \quad (f=2)$$

then $c^2 = 4.61 \qquad c = 2.15$

let ϵ = 90% contour ellipse with semi-axes $(2.15\sigma_x, 2.15\sigma_y)$

i.e. for ϵ = 90% the ellipse is $4.3\sigma_x$ by $4.3\sigma_y$, centered at a_o.

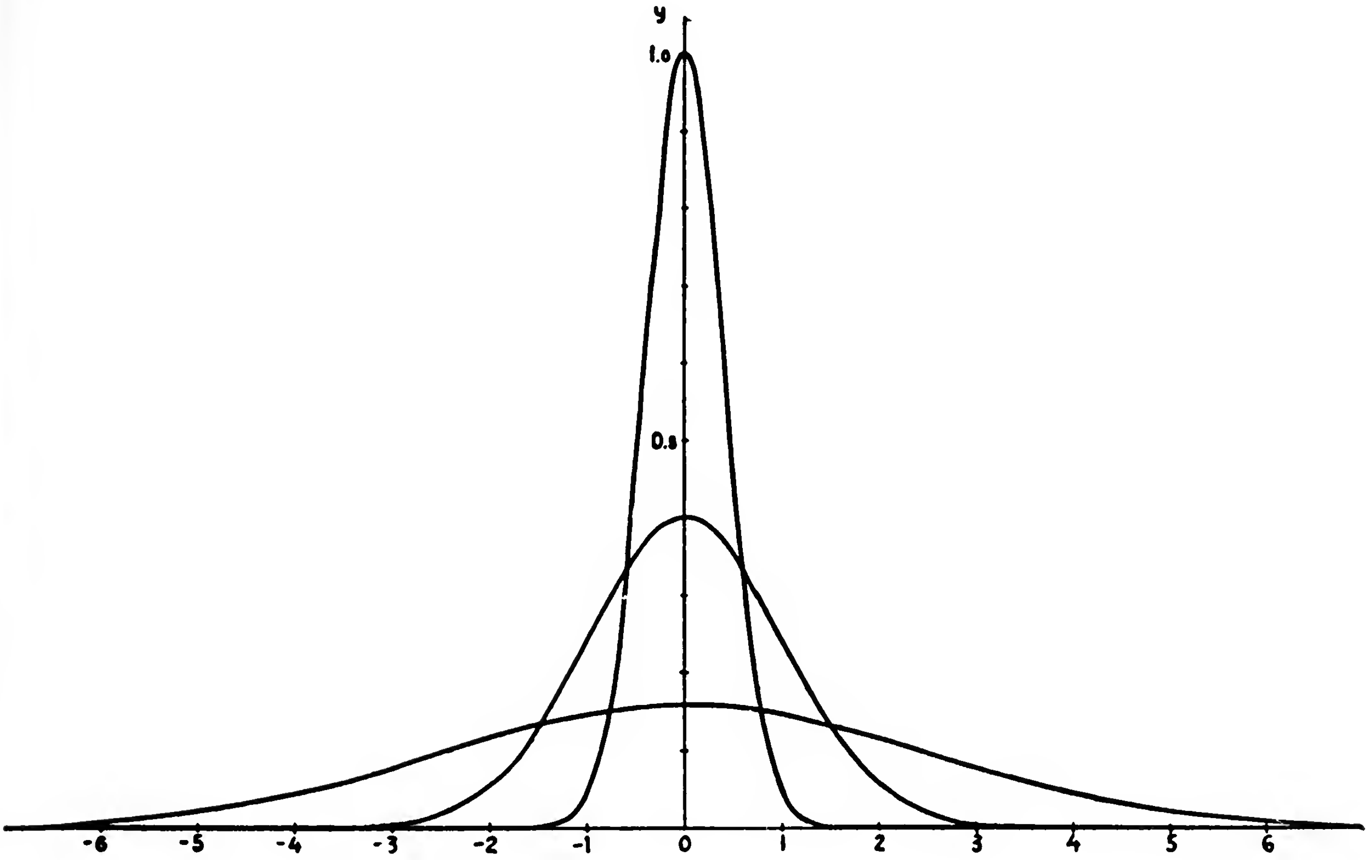


Figure A-1: Normal Frequency Curves.
m=0; σ = 0.4, 1.0, 2.5

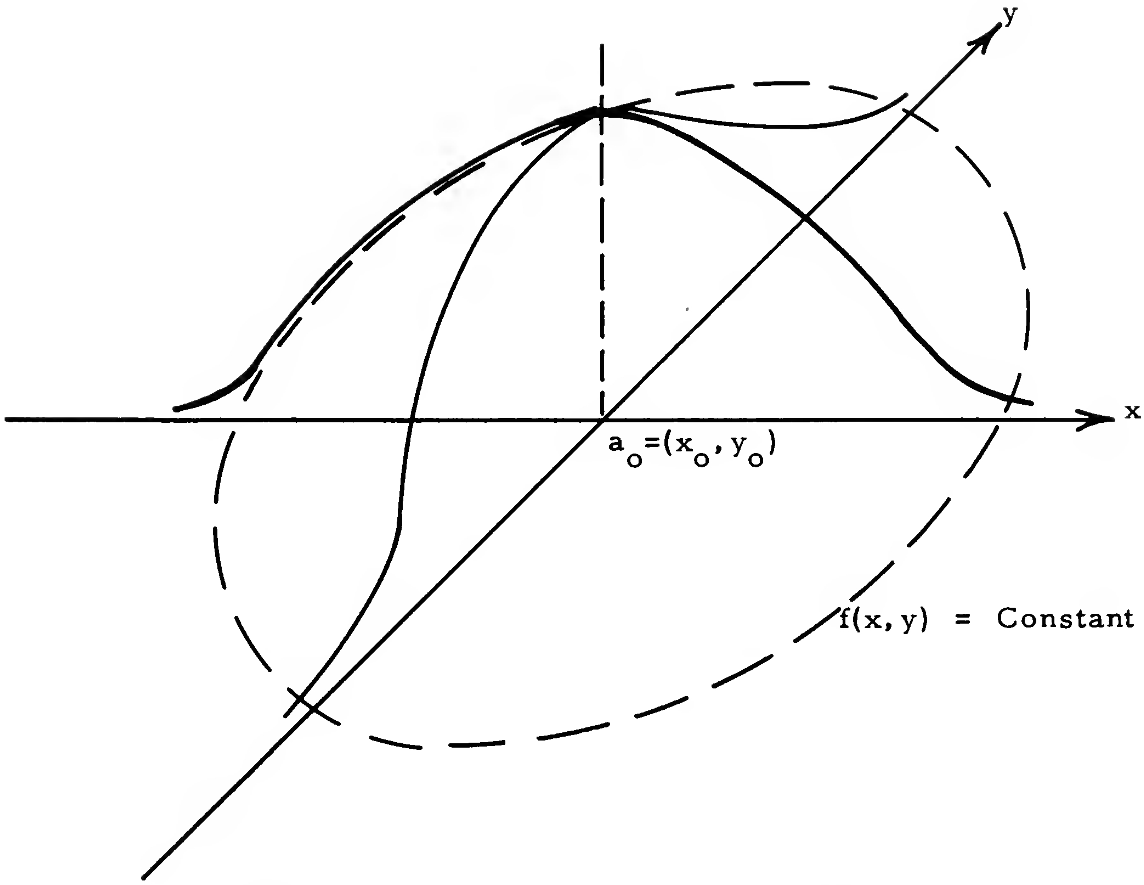


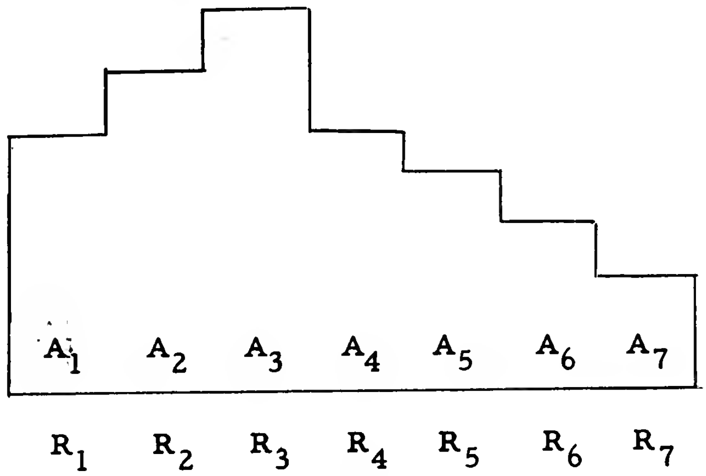
Figure A - 2: Bivariate Normal Distribution

Appendix B

Proposal for Preparation of Histograms Used in Salvo Firing

A proposal for using an electronic digital computer for determining D and F(D) suitable for use as reference diagrams.

First, prepare statistically, the histograms representing R, f(R); B, f(B); θ, $f(\theta)$ where $\theta = \theta(C, B)$; and S, f(S). Thus for R, we might have the histogram shown in the following figure.



For each component the machine is instructed to pick a random number (by a sub-routine method) say $0 \leq r \leq 1$ and this number instructs the machine from which step of the cumulative graph to pick the value of R. e.g., $0 \leq r \leq A_1$ picks R_1, $A_1 < r \leq A_1 + A_2$ picks R_2, etc. In this manner, a value of each component is picked and the machine computes the simple trigonometry involved to get the X, Y coordinates, say from the datum point for a given τ.

These are compared to the grid values, sorted, and added to the proper cell as a simple count of the occurrence.

	x_1	x_2	x_3	x_4	x_5
y_1					
y_2				$f(x_4 y_2)$	
y_3					
y_4					
y_5					

Suggest 100 yard boxes

D and f(D) for various set of components and Υ's could be make up to guide the approach officer in the selection of the p_i's and n .

Not only would one like histograms based on various mean values to use in the computation, but also on the method of measurement, such as range histograms at various ranges taken by: radar, optical, sonar, etc: Bearing: optical, hydrophone, sonar; Speed; screw count; tracking, etc.

If each histogram was made up in standard steps the computing routine could be used over and over again to construct D's and f(D) for various combinations by just changing the values in the cumulative histogram cells.

Note that the computer would produce discrete mounds that would be as good as the histogram data from which they were prepared.

No mention of what the component error data actually looks like is made here, because of the non-classified nature of the paper.

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Thesis
T825
Tucker
Submarine firing phase decision.